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New Coarse Grid Operators for Highly Oscillatory Coefficient Elliptic Problems

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New coarse grid operators are developed for elliptic problems with highly oscillatory coefficients. The new coarse grid operators are constructed directly based on the homogenized differential operators or hierarchically computed from the finest grid. A detailed description of this construction is provided. Numerical calculations for a two-dimensional elliptic model problem show that the homogenized form of the equations is very useful in the design of coarse grid operators for the multigrid method. A more realistic problem of heat conduction in a composite structure is also considered.

1. INTRODUCTION

The multigrid method is usually not effective when applied to problems for which the standard coarse grid operators have significantly different properties from those of the fine grid operators [1, 3, 6, 9, 10]. For some of these problems, in order to restore the high efficiency of the multigrid method, the coarse grid operators must be constructed on other principles than just simply restricting from the finest grid. Elliptic and parabolic equations with strongly variable coefficients and some hyperbolic equations are such problems. A common feature of these problems is that the smallest eigenvalues in absolute value do not correspond to very smooth eigenfunctions. It is thus not easy to represent these eigenfunctions on the coarser grids.

We consider two-dimensional elliptic equations with oscillatory coefficients,

$$-\nabla \cdot a^\varepsilon(x, y)\nabla u_\varepsilon = f(x, y),\quad (x, y) \in \Omega = [0, 1] \times [0, 1]. \tag{1.1}$$

Here, $a^\varepsilon(x, y) = a(x/\varepsilon, y/\varepsilon)$ is strictly positive. Because of the small parameter ε, the coefficients oscillate. The parameter ε represents the length of oscillation. A generalization of this type of equations to high dimensional space is given by

$$\sum_j \frac{\partial}{\partial x_j} a_j\left(x, \frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} u_\varepsilon(x) = f(x), \tag{1.2}$$

with $a_j(x, \eta)$ strictly positive, continuous, and 1-periodic in each component of η. For these equations, there exists a fairly complete analytic theory, known as the homogenization theory, such that a rigorous treatment is possible. The homogenization theory describes the dependence of the large scale features in the solutions from the smaller scales in the coefficients [2]. We consider problems on the form (1.1) and there are important practical applications of similar equations in the study of elasticity and heat conduction for composite materials.

By introducing new coarse grid operators, we analyze the multigrid method for an equation of type (1.1). These new operators are based on either local or analytic homogenized operators of the equation and can be numerically calculated from the finest grid operator by solving a so-called cell problem [2]. This approach can be applied in principle to more general cases.

The rest of the paper is organized as follows. The model problem and homogenization theory are introduced in Section 2. The multigrid method is briefly outlined and a detailed description of how to construct our new coarse grid operators is provided in Section 3. Numerical experiments are given in Section 4 and a general conclusion is given in Section 5, where a short discussion about convergence theory is also presented.

2. DISCRETIZATION

2.1. Partial Differential Equations

We consider two-dimensional elliptic equations (1.1) subject to Dirichlet boundary condition

$$u_\varepsilon \,|\, \partial\Omega = 0.$$

From the homogenization theory [2], it follows that

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$$\max_{(x,y)\in\Omega} |u_\varepsilon - u| \to 0, \quad \text{as } \varepsilon \to 0,$$

where u satisfies the following homogenized equation that does not contain any oscillatory coefficient,

$$-A_{11}\frac{\partial^2 u}{\partial x^2} - (A_{12} + A_{21})\frac{\partial^2 u}{\partial y \partial x} - A_{22}\frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \tag{2.1}$$

subject to the same boundary condition. The constant coefficients in (2.1) can be calculated from

$$A_{ij} = \frac{1}{|\Omega|}\int_\Omega a(s_1, s_2)\left(\delta_{ij} - \frac{\partial \kappa^j}{\partial s_i}\right) ds_1\, ds_2, \quad i, j = 1, 2, \tag{2.2}$$

where $A_{12} = A_{21}$ [12], and the auxiliary periodic functions κ^j are given by

$$-\nabla_s \cdot a(s_1, s_2)\nabla_s \kappa^j = -\frac{\partial a(s_1, s_2)}{\partial s_j}, \quad j = 1, 2. \tag{2.3}$$

Derivation of (2.1) is based on the asymptotic form

$$u_\varepsilon = u + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots,$$

followed by inserting this expansion to (1.1) and then equating the coefficients of equal powers of ε. The details are provided in [2]. For a model problem of type (1.1) with diagonally oscillatory coefficient,

$$a^\varepsilon(x, y) = a\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) = g\left(\frac{x - y}{\varepsilon}\right), \tag{2.4}$$

and the homogenized equation has the simple form

$$-\frac{(\mu + \bar{a})}{2}\frac{\partial^2 u}{\partial x^2} + (\mu - \bar{a})\frac{\partial^2 u}{\partial x\, \partial y} - \frac{(\mu + \bar{a})}{2}\frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \tag{2.5}$$

where $\mu = m(1/a^\varepsilon)^{-1}$ and $\bar{a} = m(a^\varepsilon)$ represent the harmonic average and the arithmetic average of coefficient $a^\varepsilon(x, y)$, respectively. Define the mean value $m(g)$ of an ε-periodic function $g(x)$ by

$$m(g) = \frac{1}{\varepsilon}\int_0^\varepsilon g(x)\, dx.$$

2.2. Finite Difference Equation

We discretize the domain Ω into $N \times N$ equal cells with $(N - 1) \times (N - 1)$ grid points by taking grid step size h in both the x and y directions to be $1/N$. N is chosen to make h of the same order as ε. Furthermore, as in [4, 6, 7, 10], we assume that the ratio of ε to the grid size h is an irrational number. This assumption is needed in order to guarantee convergence of the difference operator to the differential operator in (1.1) [4].

Denote $x_i = ih$, $y_j = jh$, and

$$a^h_{i,j} = a^\varepsilon\left(x_i - \frac{h}{2}, y_j\right),$$

$$b^h_{i,j} = a^\varepsilon\left(x_i, y_j - \frac{h}{2}\right),$$

$$f^h_{i,j} = f\left(x_i, y_j\right),$$

for $i, j = 1, \ldots, N$. The standard 5-point finite difference equation of (1.1) at the h-grid level is

$$-D^i_+ a^h_{i,j} D^i_- u^h_{i,j} - D^j_+ b^h_{i,j} D^j_- u^h_{i,j} = f^h_{i,j}, \tag{2.6}$$

where D^i_+ and D^i_- are the standard forward and backward divided differences in the x direction, respectively. Similarly, D^i_+ and D^j_- are ones in y direction.

For every j ($j = 1, \ldots, N - 1$), define a tridiagonal matrix A^h_j and a diagonal matrix B^h_j by

$$A^h_j = [-a^h_{i-1,j}, a^h_{i-1,j} + a^h_{i,j} + b^h_{i,j} + b^h_{i,j-1}, -a^h_{i,j}]_{i=1,\ldots,N-1},$$
$$B^h_j = [-b^h_{i,j}]_{i=1,\ldots,N-1}.$$

Expressed in vector notation, (2.6) can be rewritten as

$$L_h U_h = F_h, \tag{2.7}$$

where

$$U_h = (u^h_{1,1}, u^h_{2,1}, \ldots, u^h_{N-1,1}, \ldots, u^h_{1,N-1}, u^h_{2,N-1}, \ldots, u^h_{N-1,N-1})^{\mathrm{T}},$$
$$F_h = (f^h_{1,1}, f^h_{2,1}, \ldots, f^h_{N-1,1}, \ldots, f^h_{1,N-1}, f^h_{2,N-1}, \ldots, f^h_{N-1,N-1})^{\mathrm{T}},$$

and L_h is a block-tridiagonal matrix given by

$$L_h = \frac{1}{h^2}[B^h_{j-1}, A^h_j, B^h_j]_{j=1,\ldots,N-1}. \tag{2.8}$$

3. THE MULTIGRID METHOD

3.1. The Algorithm

Applications of the two-level multigrid method to Eq. (2.7) at the nth iteration usually take the following three steps:

1. Presmoothing step. Compute an approximation $U_h^{n+1/2}$ by applying γ_1 steps of a given iteration method to (2.7) with initial value U_h^n on the fine h-grid level. For convenience, we introduce the notation:

$$U_h^{n+1/2} = S^{\gamma_1}(U_h^n, L_h, f_h);$$

2. Coarse grid correction step. Introduce a coarse H-grid level and define a coarse grid operator L_H on this level, then

- restrict the residual to the coarse H-grid level: $d_H = I_h^H(f_h - L_h U_h^{n+1/2})$,
- solve the correction e_H: $L_H e_H = d_H$,
- update the approximation by interpolating the correction back to the h-grid level: $\tilde{U} = U_h^{n+1/2} + I_H^h e_H$;

3. Postsmoothing step. Repeat step 1 with the approximation from step 2 as the initial value,

$$U_h^{n+1} = S^{\gamma_2}(\tilde{U}, L_h, f_h).$$

The iteration operator M of the two-level multigrid method is thus given by

$$M = S^{\gamma_2}(I - I_H^h L_H^{-1} I_h^H L_h) S^{\gamma_1}. \tag{3.1}$$

For the full multigrid method, the correction in step 2 is solved by applying the two-level multigrid method recursively. The same procedure can be repeated several times until the coarest level is reached, where the correction equation is solved exactly. We always take the current coarse grid step size to be twice as big as the preceding one. Let $V(\gamma_1, \gamma_2)$ denote the full multigrid cycle with γ_1 steps as the presmoothing and γ_2 steps as the postsmoothing on all levels.

3.2. Construction of Coarse Grid Operators

By the homogenized equation and the asymptotic behavior of the associated eigenvalue problem [1, 2, 7, 9, 10], one can show that the small eigenvalues of the original oscillatory operator can be approximated by the corresponding homogenized eigenvalues. After a few steps of fine grid smoothing, the error will be dominated by the low frequency modes and these modes can be approximated by the corresponding homogenized ones at the coarser grid level. Based on this idea, we construct the coarse grid operator directly from the homogenized operator. Three different techniques are described below.

The *analytic homogenized coarse grid operator* is a discretization of the analytic form (2.1) of the homogenized operator. This requires the analytic form to be known or to be computed a priori. If the homogenized equation has a simple analytic form the effective coefficients following this form can be approximated locally and therefore adjust better to local variations. We denote this technique *local homogenized coarse grid operator*. Finally we describe how to derive the *local numerically homogenized coarse grid operator*. This is the most general approach and can, as a procedure, be applied to any elliptic problem; compare [5]. The effective coefficients are computed locally based on the solution of a cell problem and this can be seen as a direct extension of the *local homogenized coarse grid operator* mentioned above.

3.2.1. *Analytic Homogenized Coarse Grid Operator*

If we construct the coarse grid operator directly from the discretization of the corresponding homogenized operator in (2.1), we obtain an *analytic homogenized coarse grid operator* L_H at H-grid level,

$$\begin{aligned} L_H = [&-A_{11} D_+^i D_-^i - A_{22} D_+^j D_-^j \\ &- \theta(A_{12} + A_{21}) D_0^i D_0^j]_{i,j=1,\ldots,1/H-1}, \end{aligned} \tag{3.2}$$

where D_0 denotes the standard center divided difference and θ is a parameter. For (2.4), L_H can be simplified as

$$\begin{aligned} L_H = \Bigg[&-\frac{(\mu + \bar{a})}{2} D_+^i D_-^i - \frac{(\mu + \bar{a})}{2} D_+^j D_-^j \\ &+ \theta(\mu - \bar{a}) D_0^i D_0^j \Bigg]_{i,j=1,\ldots,1/H-1}, \end{aligned} \tag{3.3}$$

where μ, $\bar{a}$, and θ have constant values through the domain.

3.2.2. *Local Homogenized Coarse Grid Operator*

In order to better approximate the fine grid operator, we construct a *local homogenized coarse grid operator* using the homogenized operator with coefficients generated locally. To do this, we first divide the entire domain into many cells. For instance, at point (i, j) on the coarse H-grid level (see Figs. 1 and 2) we may define four cells, denoted by EH, WH, SH, NH. In each cell, we calculate a homogenized operator as in (2.1) with coefficients determined in this cell. We then derive a coarse grid operator L_H on H-grid level that maintains the form of the homogenized operator but with variable coefficients. L_H is of the form

$$\begin{aligned} L_H = = [&-D_+^i a_{i,j}^H D_-^i - D_+^j b_{i,j}^H D_-^j - \theta D_0^i c_{i,j}^H D_0^j \\ &- \theta D_0^j c_{i,j}^H D_0^i]_{i,j=1,\ldots,1/H-1}. \end{aligned} \tag{3.4}$$

When the analytic homogenized operator has a simple form the coefficients can be directly calculated from local values. For the model problem (2.4), the form is given by (2.5) and the coefficients can be determined as follows.

The value of $a_{i,j}^H$ is the coefficient of $\partial^2 u/\partial x^2$ in the homog-

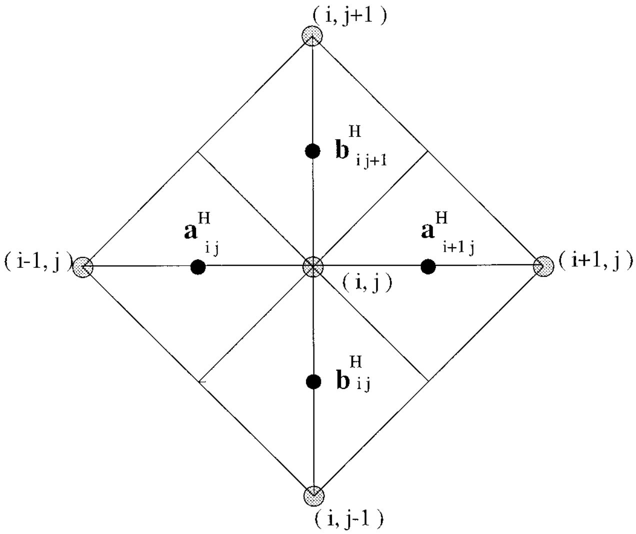

FIG. 1. Coefficients for H-grid level at (i, j).

enized equation (2.1) generated in a WH-cell by (2.2). For (2.4),

$$a^H_{i,j} = \tfrac{1}{2}(\mu(WH) + \overline{a}(WH)),$$

where $\overline{a}(WH)$ and $\mu(WH)$ denote the arithmetic average and the harmonic average of all $a^h_{i,j}, b^h_{i,j}$ in WH-cell, respectively.

The value of $b^H_{i,j}$ is the coefficient of $\partial^2 u/\partial y^2$ in the homogenized equation (2.1) generated in a SH-cell by (2.2). For (2.4),

$$b^H_{i,j} = \tfrac{1}{2}(\mu(SH) + \overline{a}(SH)),$$

where $\overline{a}(SH)$ and $\mu(SH)$ denote the arithmetic average and the harmonic average of all $a^h_{i,j}, b^h_{i,j}$ in a SH-cell, respectively.

The value of $c^H_{i,j}$ is the coefficient of $\partial^2 u/\partial x \partial y$ in the homogenized equation (2.1) generated in cells EH, NH, SH, WH by (2.2). For (2.4),

$$c^H_{i,j} = \tfrac{1}{2}(-\mu(EH, NH, SH, WH) + \overline{a}(EH, NH, SH, WH)),$$

where $\overline{a}(EH, NH, SH, WH)$ and $\mu(EH, NH, SH, WH)$ denote the arithmetic average and the harmonic average of all $a^h_{i,j}$, $b^h_{i,j}$ in EH, NH, SH, WH-cells, respectively.

The above locally discrete homogenization procedure is used for computations in this paper. For other approximative homogenization techniques we refer the reader to papers [8, 5] and the section below.

3.2.3. *Local Numerically Homogenized Coarse Grid Operator*

For equations of type (1.1), it is not always possible to obtain A_{ij} in (2.2) explicitly, and the derivation of A_{ij} usually involves numerically solving κ^j in (2.3). We extend the construction of the *local homogenized coarse grid operator* as follows. Given a H-grid level, let h now denote the fine

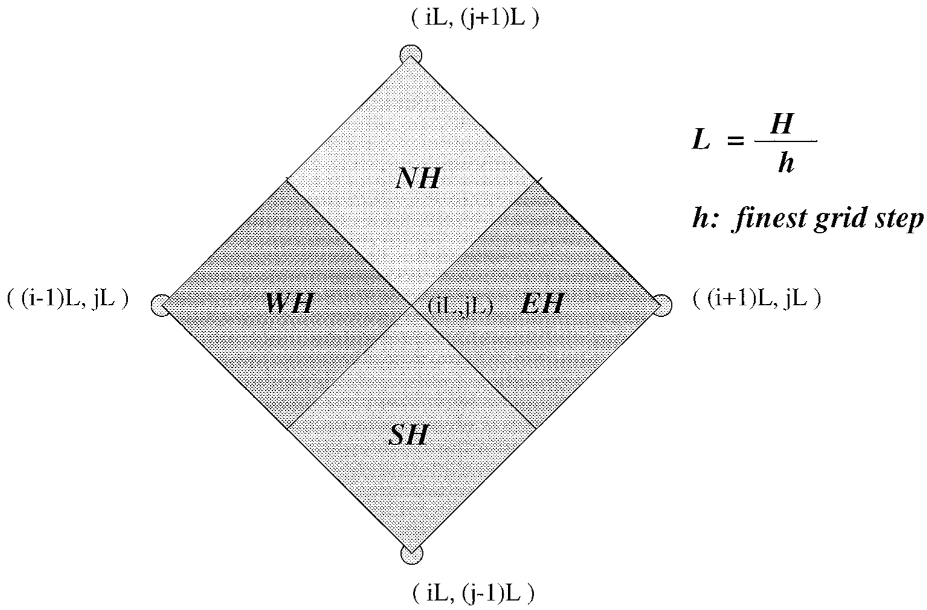

FIG. 2. Construction of 4 cells on the coarse H-grid level at point (i, j) with respect to the finest h-grid level at (iL, jL).

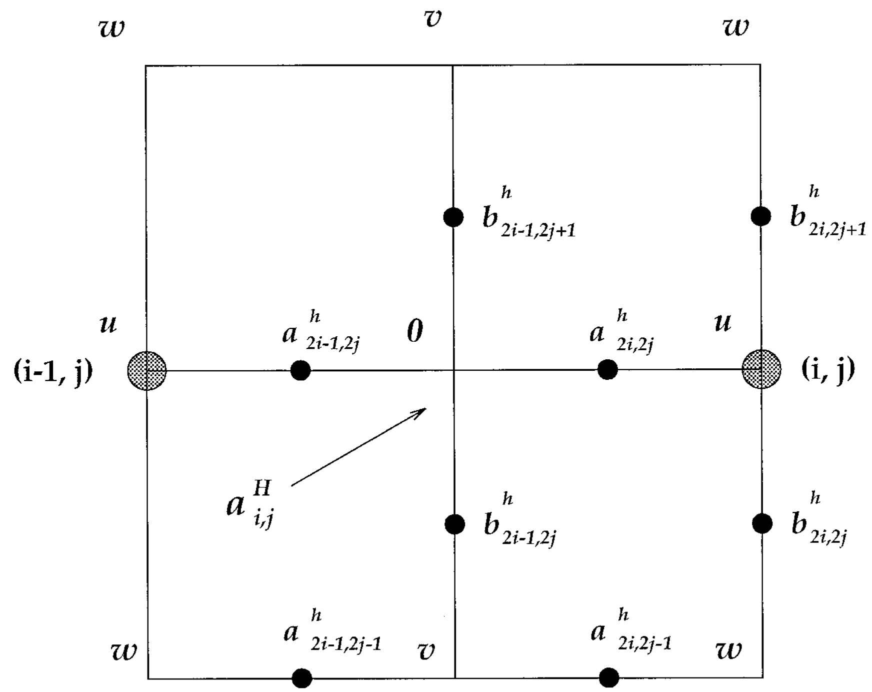

FIG. 3. Cell for the construction of $a^H_{i,j}$ on the H-grid level at (i, j).

grid level that immediately precedes H and let L_h denote the operator for the correction at h-grid level defined by

$$L_h = [-D^i_+ a^h_{i,j} D^i_- - D^j_+ b^h_{i,j} D^j_- + D^i_0 c^h_{i,j} D^j_0 + D^j_0 c^h_{i,j} D^i_0]_{i,j=1,\ldots,1/h-1}. \tag{3.5}$$

For the coefficient $a^H_{i,j}$ in (3.4) at a point (i, j) on the coarse H-grid level, construct a cell as in Fig. 3. Next, define an auxiliary periodic function κ over the cell such that it takes values u, v, and w on different grid points as indicated in Fig. 3, and 0 at the center. From the homogenization theory, we establish the discretized equation for κ at the center point $(2i - 1, 2j)$ on the h-grid level by

$$\begin{aligned}(L_h\kappa)_{2i-1,2j} &= (-D^i_+ a^h_{2i-1,2j} D^i_- - D^i_0 c^h_{2i-1,2j} D^j_0 \\ &\quad - D^j_0 c^h_{2i-1,2j} D^i_0 - D^j_+ b^h_{2i-1,2j} D^j_-)\kappa_{2i-1,2j} \\ &= -D^i_+ a^h_{2i-1,2j} - D^i_0 c^h_{2i-1,2j}.\end{aligned} \tag{3.6}$$

By assuming the periodicity for $c^h_{2i-1,2j}$ over this cell, we solve (3.6) and get

$$(a^h_{2i,2j} + a^h_{2i-1,2j})u + (b^h_{2i-1,2j+1} + b^h_{2i-1,2j})v = (a^h_{2i,2j} - a^h_{2i-1,2j})h. \tag{3.7}$$

In order to establish two more relations among u, v and w, we construct two more cells as in Fig. 4. These two cells are based on the extension of the cell in Fig. 3, and the values of κ and coefficients in (3.6) at grid points are periodically taken as in Fig. 4. Similarly, from the equations for κ at points $(2i, 2j)$ and $(2i - 1, 2j - 1)$ on the h-grid level, we obtain

$$-(a^h_{2i,2j} + a^h_{2i-1,2j} + b^h_{2i,2j} + b^h_{2j+1})u + (b^h_{2i,2j} + b^h_{2i,2j+1})w = (a^h_{2i-1,2j} - a^h_{2i,2j})h, \tag{3.8}$$

$$-(a^h_{2i-1,2j-1} + a^h_{2i,2j-1} + b^h_{2i-1,2j} + b^h_{2i,2j})v + (a^h_{2i-1,2j-1} + a^h_{2i,2j-1})w = (a^h_{2i,2j-1} - a^h_{2i-1,2j-1})h. \tag{3.9}$$

From (3.7)–(3.9), we can solve u, v, and w. Based on the analytic formula [2], we construct the discrete coefficient $a^H_{i,j}$ on the H-grid level in the cell in Fig. 3 as

$$a^H_{i,j} = \tfrac{1}{2}\left(a^h_{2i-1,2j} + a^h_{2i,2j} + \frac{(a^h_{2i-1,2j} - a^h_{2i,2j})}{h}u\right).$$

We can construct $b^H_{i,j}$ similarly using the cells and auxiliary parameters indicated in Fig. 5,

$$b^H_{i,j} = \tfrac{1}{2}\left(b^h_{2i,2j-1} + b^h_{2i,2j} + \frac{(b^h_{2i,2j-1} - b^h_{2i,2j})}{h}v\right),$$

by solving v from

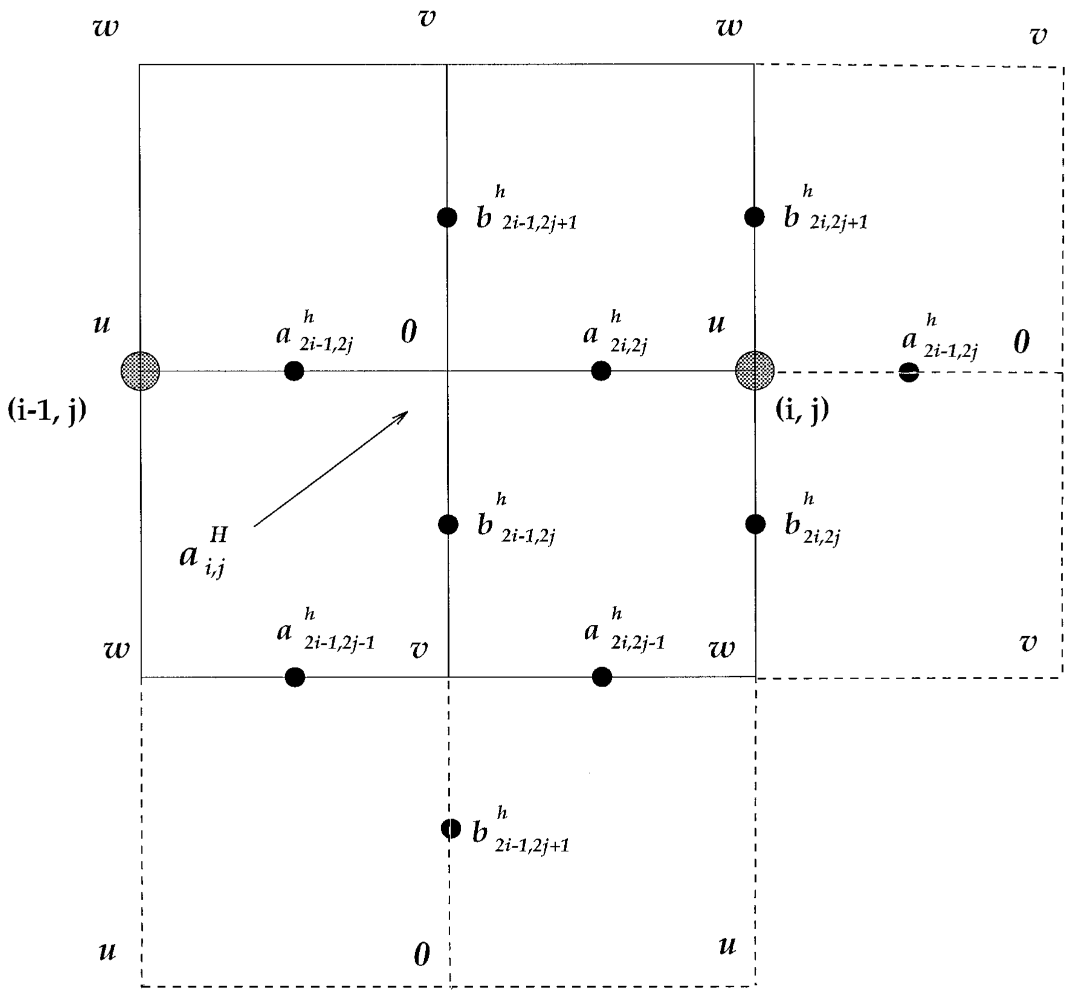

FIG. 4. Auxilliary cells for the construction of $a^H_{i,j}$ on the H-grid level at point (i, j).

$$(L_h\kappa)_{2i,2j-1} = -D^j_+ b^h_{2i,2j-1} - D^j_0 c^h_{2i,2j-1},$$

$$(L_h\kappa)_{2i,2j} = -D^j_+ b^h_{2i,2j} - D^j_0 c^h_{2i,2j},$$

$$(L_h\kappa)_{2i-1,2j-1} = -D^j_+ b^h_{2i-1,2j-1} - D^j_0 c^h_{2i-1,2j-1}.$$

For $c^H_{i,j}$ at point (i, j) on the H-grid level, we first construct a cell consisting of four subcells NW, NE, SW, SE as indicated in Fig. 6. In each subcell, we solve κ in the same way as before. Based on the analytic formula [2] for $c^H_{i,j}$, we construct the discretized $c^H_{i,j}$ over the entire cell in Fig. 6 by

$$\begin{aligned} c^H_{i,j} = \tfrac{1}{16}(&4c^h_{2i-1,2j} + (b^h_{2i-1,2j} - b^h_{2i-1,2j+1})v_{sw} \\ &+ (a^h_{2i-1,2j} - a^h_{2i,2j})u_{sw} + 4c^h_{2i,2j-1} \\ &+ (b^h_{2i,2j-1} - b^h_{2i,2j})v_{se} + (a^h_{2i,2j-1} - a^h_{2i+1,2j-1})u_{se} \\ &+ 4c^h_{2i+1,2j} + (b^h_{2i+1,2j} - b^h_{2i+1,2j+1})v_{nw} \\ &+ (a^h_{2i+1,2j} - a^h_{2i+2,2j})u_{nw} + 4c^h_{2i,2j+1} \\ &+ (b^h_{2i,2j+1} - b^h_{2i,2j+2})v_{ne} + (a^h_{2i,2j+1} - a^h_{2i+1,2j+1})u_{ne}) \end{aligned}$$

where v_{sw} is solved by the construction for $a^H_{i,j}$, and u_{sw} by the construction for $b^H_{i-1/2,j+1/2}$ over cell SW. Similarly, v_{se}, u_{se}, v_{nw}, u_{nw}, v_{ne} & u_{ne} are solved on the different subcells, respectively.

3.3. Construction of Interpolation

We consider a harmonic interpolation I^h_H in this paper. By the continuity of $a^\varepsilon(x, y)\,(\partial u_\varepsilon/\partial x)$ and $a^\varepsilon(x, y)\,(\partial u_\varepsilon/\partial y)$, such an interpolation can be constructed as follows (see [1]). Set

$$a^h_{2i-1,2j} D^i_- u^h_{2i-1,2j} = a^h_{2i,2j} D^i_- u^H_{2i,2j},$$

$$b^h_{2i,2j-1} D^j_- u^h_{2i,2j-1} = b^h_{2i,2j} D^j_- u^H_{2i,2j},$$

and then solve at points $(2i-1, 2j)$ & $(2i, 2j-1)$

$$u^h_{2i-1,2j} = \frac{a^h_{2i-1,2j} u^H_{2i-2,2j} + a^h_{2i,2j} u^H_{2i,2j}}{a^h_{2i-1,2j} + a^h_{2i,2j}};$$

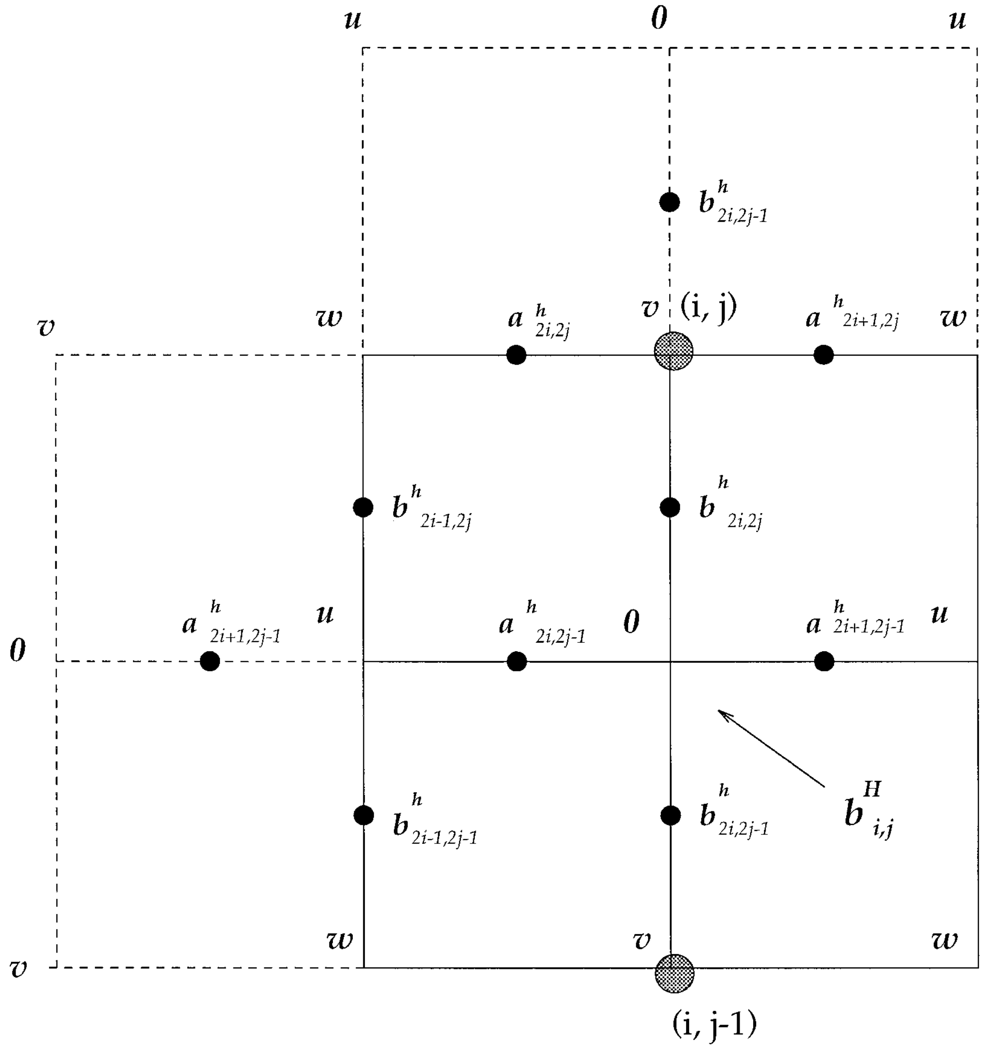

FIG. 5. Cells for the construction of $b^H_{i,j}$ on the H-grid level at point (i, j).

$$u^h_{2i,2j-1} = \frac{b^h_{2i,2j-1} u^H_{2i,2j-2} + b^h_{2i,2j} u^H_{2i,2j}}{b^h_{2i,2j-1} + b^h_{2i,2j}}.$$

At point $(2i - 1, 2j - 1)$, use the following weighted interpolation,

$$u^h_{2i-1,2j-1} = \frac{a^h_{2i-1,2j-1} u^h_{2i-2,2j-1} + a^h_{2i,2j-1} u^h_{2i,2j-1} + b^h_{2i-1,2j-1} u^h_{2i-1,2j-2} + b^h_{2i-1,2j} u^h_{2i-1,2j}}{a^h_{2i-1,2j-1} + a^h_{2i,2j-1} + b^h_{2i-1,2j-1} + b^h_{2i-1,2j}},$$

where $i, j = 1, ..., N/2 - 1$.

For the restriction operator I^H_h, we take it to be the transpose $I^H_h = (I^H_h)^{\mathrm{T}}$ of the prolongation.

4. NUMERICAL RESULTS

In this section, the multigrid method with the homogenized coarse grid operators and harmonic interpolation constructed in last section is applied to two examples. We use these examples to study convergence property of the $V(\gamma_{\mathrm{i}}, \gamma_2)$-cycle multigrid method. For this purpose, we consider the mean rate ρ of convergence of the method, where ρ is defined by (see [13])

$$\rho = \left(\frac{\|L_h u^i - f_h\|_h}{\|L_h u^1 - f_h\|_h} \right)^{1/(i-1)}, \tag{4.1}$$

where i is the smallest integer satisfying $\|L_h u^i - f_h\|_h \leq 1 \times 10^{-9}$ and $\|\cdot\|_h$ denotes the discrete l_2 norm.

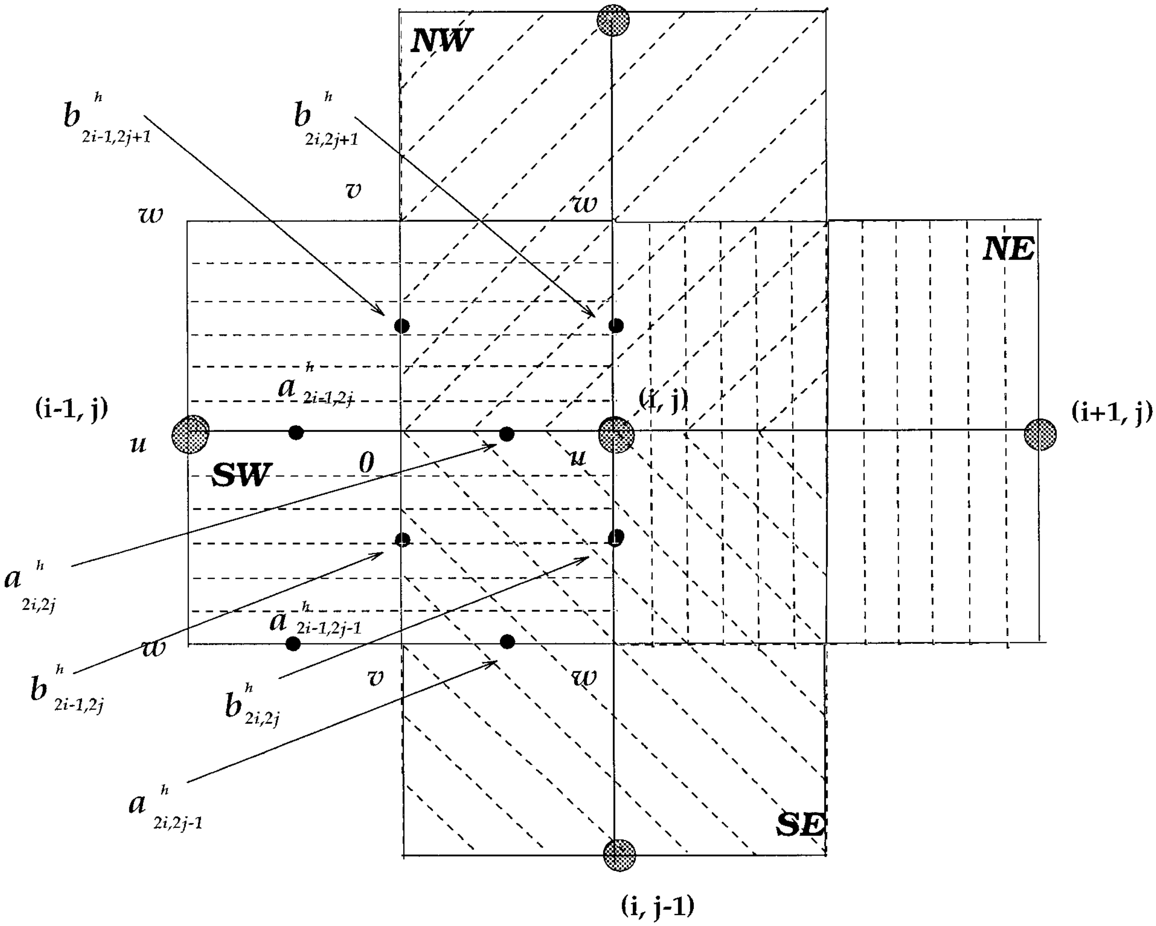

FIG. 6. Cells for the construction of $c_{i,j}^{H}$ on the H-grid level at point (i, j).

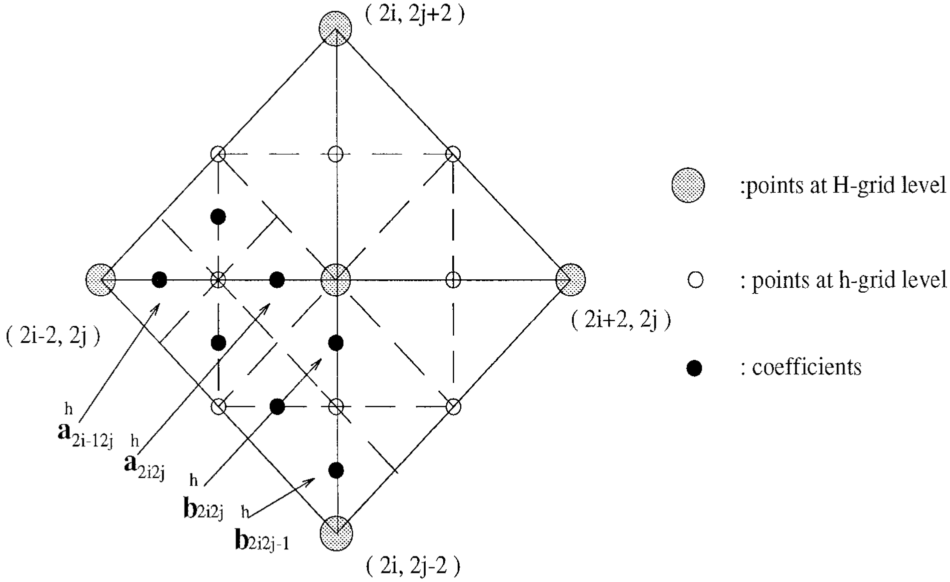

FIG. 7. Interpolation from H-grid level to h-grid level.

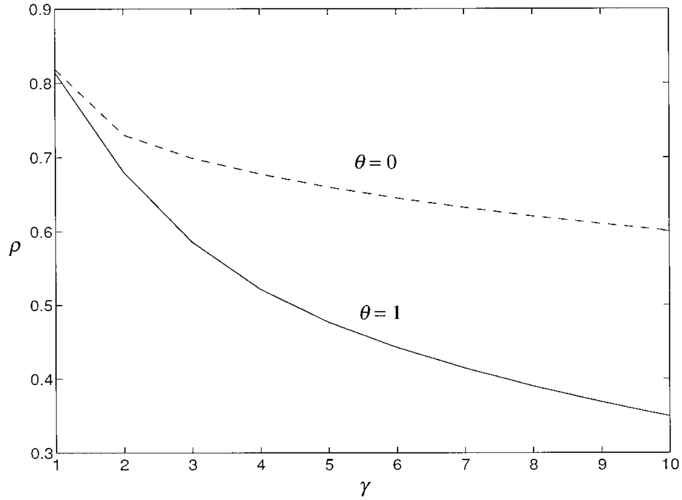

FIG. 8. Spectral radius ρ as a function of smoothing step γ.

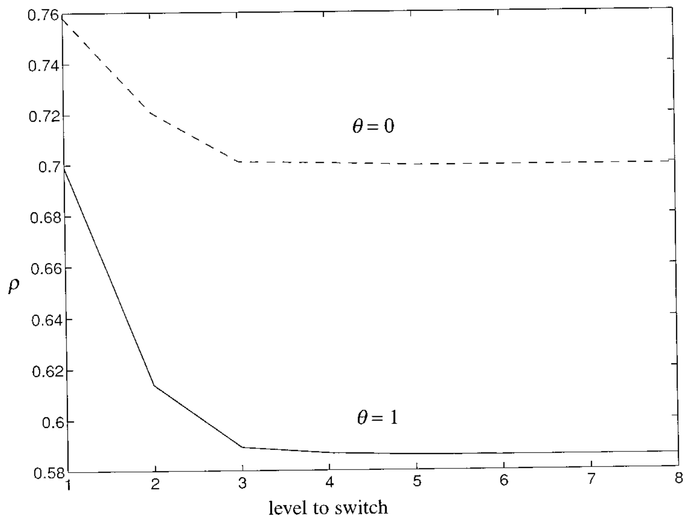

FIG. 9. Spectral radius ρ as a function of a level variable for $V(3, 3)$.

4.1. Example 1: Model Problem

In the following numerical experiments, unless otherwise noted, we consider Eq. (1.1) with coefficient $a^{\varepsilon}(x, y) = 2.1 + 2\sin(2\pi(x - y)/\varepsilon)$, where the harmonic average μ and arithmetic average $\overline{a}$ are given by

$$\mu = 0.64, \quad \overline{a} = 2.1.$$

The smoothing iteration operator S is based on the following damped Jacobi iteration,

$$S = I - \omega h^2 L_h. \tag{4.2}$$

The finest grid points are on a 256×256 mesh and the step at the finest grid level h has an irrational relation with ε, i.e., $\varepsilon = \sqrt{2}h$. The step size of the coarest level equals $\frac{1}{2}$, and ω in (4.2) is 0.095 which is tested numerically to be the best. In the numerical experiments, we compare two cases corresponding to two different values for parameter θ introduced in (3.2) and (3.4). When $\theta = 1$, the coarse grid operator is the homogenized operator; when $\theta = 0$, the coarse grid operator is the operator in (3.2) and (3.4) without cross terms. In the latter case, the coarse grid operator is no longer the homogenized operator.

For Fig. 8, we apply the *local homogenized coarse grid operator* (3.4) to all coarse grid levels. In this figure, the spectral radius ρ for $V(\gamma, \gamma)$ is plotted against the smoothing step γ. Notice that the rate of convergence for the multigrid method is faster for $\theta = 1$ than that for $\theta = 0$. This observation becomes clearer as the smoothing step gets larger. In fact, after a few coarse grid levels, the *local homogenized coarse grid operator* can be replaced with the *analytic homogenized coarse grid operator* (3.2). This means that we only need to apply the *local homogenized coarse grid operator* to a few first coarse grid levels, and then apply the *analytic homogenized coarse grid operator* to the remaining coarse grid levels. In Fig. 9, we plot the spectral radius as a function of a level variable beyond which we switch from the *local homogenized coarse grid operator* to the *analytic homogenized coarse grid operator*. This figure shows roughly how the process of eigenmodes of error in the multigrid method can be reduced. It is clear that, after the finest grid level smoothing, there do exist many intermediate eigenmodes which are not quite close to the homogenized ones. In Fig. 10 we plot the spectral radius as a function of variable θ for $V(3, 3)$. From this figure, we can see as θ goes to 1, the convergence of the

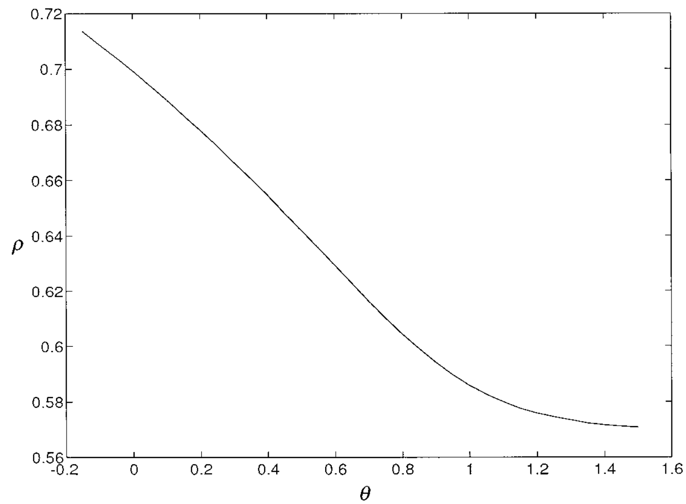

FIG. 10. Spectral radius ρ as a function of variable θ.

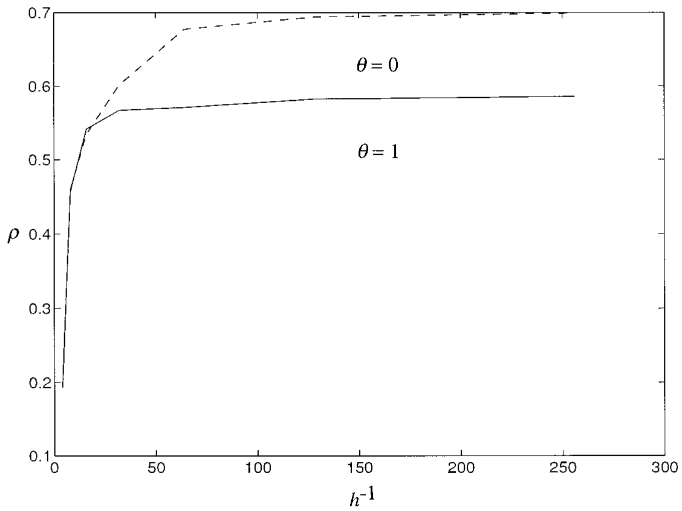

FIG. 11. Spectral radius ρ as a fuunction of $1/h$.

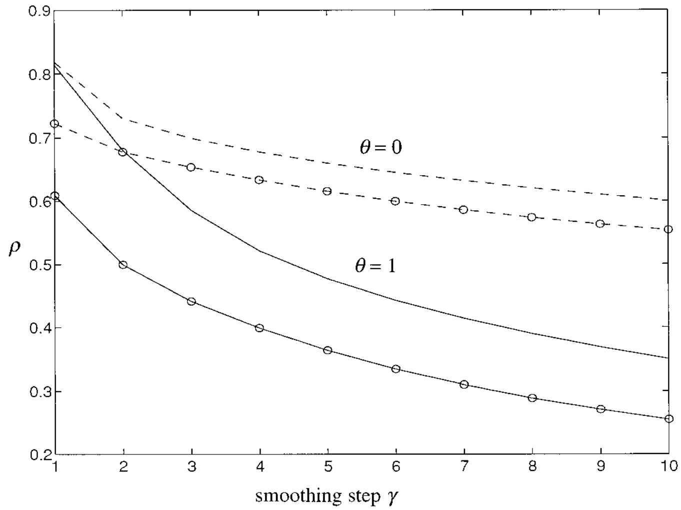

FIG. 12. Spectral radius ρ as a function of smoothing steps γ for $V(\gamma, \gamma)$. Lines with circle for SOR iteration; otherwise, for Jacobi.

multigrid method can be much improved. In Fig. 11, we plot the spectral radius as a function of $1/h$ for $V(3, 3)$, where h is the grid step size of the finest level. As shown in this figure, the convergence rate of the multigrid method does depend on the grid size h.

To summarize, we have shown by numerical experiments in various aspects that the rate of convergence of the multigrid method can be much improved with the homogenized operator as the coarse grid operator. Up to this point, in order to isolate the influence of the coarse grid approximation we have kept the smoothing operator fixed. If we use SOR iteration method in (4.2), the convergence rate can be further improved. We compare the convergence rate by choosing damped Jacobi iteration and SOR iteration in Fig. 12.

4.2. Example 2: Application

To show the extension of using the homogenized operator as the coarse grid operator in the multigrid method to more general cases (e.g., ones involving discontinuous coefficients) we consider a practical problem described in Fig. 13 below. The problem can be viewed as a wall with a composite material for insulation in the center. We are interested in the heat conduction in such a composite structure. The governing equation has the form of (1.1) and is given by

$$-\frac{\partial}{\partial x} C(x, y) \frac{\partial U}{\partial x} - \frac{\partial}{\partial y} C(x, y) \frac{\partial U}{\partial y} = 100,$$

in a rectangular domain $(x, y) \in \Omega = (0, 1) \times (0, 2)$. Here $C(x, y)$ is the conductivity parameter. Boundary conditions and other parameters are given in Fig. 13. Although the coefficient does not satisfy the periodic assumption which is needed in homogenization theory, it preserves some essential property as before from a probablistic point of view. Namely, it is highly oscillatory in the middle part of the domain. Standard discretization of the equation contains almost randomly distributed magnitude coeffi-

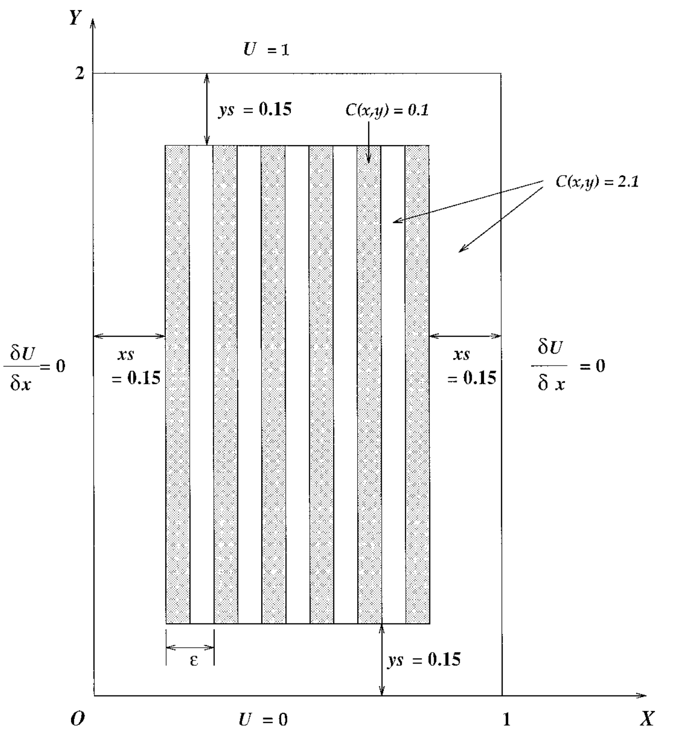

FIG. 13. Model.

TABLE I

Spectral Radius ρ

L	3	4	5	6	7	8
Method 1	0.1514	0.3779	0.4005	0.3922	0.4752	0.4832
Method 2	0.1416	0.3084	0.5168	0.5518	0.6358	0.7066
Method 3	0.4596	0.7028	0.8843	0.9595		

cients by taking $\varepsilon = \sqrt{2}h$. In such a case, the coarse grid operators can still be generated by a similar idea as introduced in the previous section. Since the conductivity here is strongly varying in x direction, and has two interfaces in y direction, the structure of the homogenized operator approximately has the form

$$- A_{11} \frac{\partial^2 U}{\partial x^2} - A_{22} \frac{\partial^2 U}{\partial y^2}, \tag{4.3}$$

where A_{11} and A_{22} are harmonic average and arithmetic average of the coefficient, respectively.

In Table I, we calculate the mean rate ρ of the multigrid method $V(2L, 2L)$ with the discrete coarse grid operator (4.3) and $\omega = 1.7$ for SOR. This method is referred here as Method 1. For comparison, the mean rate ρ is also calculated under the following two different standard methods:

Method 2. Multigrid method with the discrete coarse grid operator (4.3), where both A_{11} and A_{22} are generated by the local arithmetic average of the coefficients.

Method 3. Direct SOR iteration method without any coarse grid correction.

The number of grid points at the finest level is chosen to be 2^L and the grid step at the coarest level for the multigrid methods is $\frac{1}{2}$.

Among the above methods, Method 1 gives the fastest convergence rate as shown in Table I. This example shows that the multigrid method with homogenized operator as the coarse grid operator is also quite applicable to problems with discontinuous coefficients.

5. CONCLUSION

In this paper we have presented three new types coarse grid operators for the multigrid method. These new operators are called homogenized coarse grid operators. We have given a detailed description about how to construct these operators for two-dimensional elliptic problems. The constructions can be applied to more general cases, as shown by the practical example with discontinuous coefficients. The homogenized coarse grid operators improve the convergence rate of the multigrid method for elliptic equations with oscillatory coefficients. This conclusion is supported by the computations.

The convergence of the two-level multigrid method with the *analytic homogenized coarse grid operator* is analyzed in [7] for elliptic equations with Dirichlet boundary conditions. Without requiring the ratio of h to ε to be small, we prove that when both ε and h go to zero, as long as they satisfy a sampling condition, the two-level multigrid method converges if the iteration number $\gamma \geq Ch^{-\alpha} \ln h$. The exponent $\alpha = 1 + \frac{1}{3}$ or $1 + \frac{2}{3}$ depending on the problems.

The theoretical proof indicates the role of the homogenized operator in the convergence analysis for the multigrid method. Results of numerical experiments show that faster convergence rate in practice can be achieved than that guaranteed by theoretical results. However, numerical results do indicate that the convergence rate depends on the grid size h for equations with oscillatory coefficients.

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